SENIOR MATHEMATICS OLYMPIAD-2025

Answer all the questions. Symbols carry their usual meaning.
Each of the questions carry 5 marks

Time- 3 Hours

Full Marks- 100

1. Show that $\frac{a(a^2+2)}{3}$ is an integer for any integer.
2. Find all bijective functions f on $\mathbb{Q}$ such that f satisfies $f(p+q) = f(p) + f(q)$ and $f(pq) = f(p)f(q)$ for any p, q in $\mathbb{Q}$.
3. Let H be a subgroup of a group G. If $x^2 \in H$ for all $x \in G$, then prove that H is a normal subgroup of G and G/H is commutative.
4. Let $T: \wp_2(\mathbb{R}) \rightarrow \wp_4(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$, where $\wp_k(\mathbb{R})$ denotes the set of all polynomials of degree less than or equal to k, for $k = 2,4$ .Find the range of T, rank of T. Check whether T is one to one and onto.
5. Let A be a 2×2 real matrix such that $AA^T = I$. If $\det A = 1$, then show that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for any value of θ.
6. Let S be a map on $\mathbb{C}$ defined by $S(z) = \frac{2i(z+1)}{(1+i)(z+i)}$. Show that $S(z)$ is real if z lies in the unit circle in $\mathbb{C}$.
7. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.
8. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfies the equation $f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all x, y belonging to $\mathbb{R}$. Find the value of $\log_e f(x)$.
9. If A is an $n \times n$ matrix over $\mathbb{C}$ such that all non - zero vectors in $\mathbb{C}^n$ is an eigenvector of A, then show that all the eigenvalues of A are equal. Find the minimal polynomial satisfied by A .
10. Let R be the rectangular region$0 \leq x \leq \pi,\ 0 \leq y \leq 1$. Find the point where the function $f(z) = \sin z$ takes the maximum value in R.
11. Two players A and B participated in a game of throwing two dice. The first player who gets a sum of 7 is awarded the prize. If A starts the game, then find the probability of their winning.
12. Five friends deposited their coats in a cloak room. When they returned to claim their coats they were all mixed up. What is the probability that none gets his own coat ?
13. Show that {Sin n : n $\in \mathbb{Z}$ } is dense in the interval [-1, 1] .
14. Test the convergence of the series (i) $\sum_{n=0}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ (ii) $\sum \left(\frac{1}{1 - \frac{1}{n \log n}} \right)^n$
15. f is a continuous function in the closed interval [0, 1]. If $\int_0^1 x^n f(x)dx = 0$ for n = 1, 2., 3, ... Show that $f(x) \equiv 0$.
16. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.
17. Let f : [0,1] → $\mathbb{R}$ be defined as $f(x) = f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ \frac{1}{p} \text{ if } x = \frac{p}{q} \text{ and } (p,q) = 1 \end{cases}$, Show that f is discontinuous at every rational point but continuous at the irrational points.
18. Solve the differential equation $\frac{d^2x}{dt^2} + x = \text{constant}$.
19. Let p be a point on an ellipse. F_1 and F_2 its foci. Show that the normal to the ellipse at p bisects the angle between PF_1 and PF_2
20. Find the cube root of 2, correct to two places of decimal using Newton-Raphson Method.

***