SENIOR MATHEMATICS OLYMPIAD (2023-24)

21ST JANUARY 2024

Answer all the questions. Symbols carry their usual meaning.

Each of the questions carry 5 marks

Time-3 Hours **Full Mark-100**

1. Find the last three digits of 7^{9999}.
2. Prove that for positive real numbers a, b, c, d
$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2$$
3. Prove that there is no simple group of order 56.
4. Find a finite group G with a normal subgroup H such that |Aut H|>|Aut G|
5. Test uniform continuity of the function f(x) =sin (x sinx) on $\mathbb{R}$.
6. Let A be the 2x2 matrix $\begin{pmatrix} sin\frac{\pi}{18} & -sin\frac{4\pi}{9} \\ sin\frac{4\pi}{9} & sin\frac{\pi}{18} \end{pmatrix}$.Find the smallest number n $\in$ N such that $A^n = I$.
7. Let $G = Z \times Z \times Z$, $H =${(x,y,z) $\in G$ | x,y,z are even integers}. Is H a normal subgroup of G? Find G/H.Explain your answer.
8. If I,J are two maximal ideals in a principal ideal domain that is not a field, then show that IJ is never a prime ideal.
9. Let A, B be $n \times n$ matrices such that $A \geq B$ .Show that $A^{\frac{1}{2}} \geq B^{\frac{1}{2}}$.
10. Let n be a positive integer and let $A, B \in M_n(\mathbf{C})$ such that $A^2 = A$ and $B^2 = B$.Then show that rank(A − AB) + rank(AB − B) = rank(A − B).
11. Find the value of the integral $\oint_{|z|=1} \frac{1}{sin\, e^{\frac{1}{z}}} dz$.
12. Can you find an analytic function $f(z)$ in the disk $|z+i| < 5$ such that $f''(-i) = i$, and $\max_{|z+i|<5}|f(z)| = 5$? Explain your answer.
13. Let $\varphi: [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 \varphi(t)e^{-at}dt = 0$ for every $a \in \mathbb{R}^+$. Show that for every non-negative integer n, $\int_0^1 \varphi(t)t^n dt = 0$.
14. Solve $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to boundary condition u(0,y)=0 and u(x,1)=x^2.
15. Let $\{a_n\}$be a recursive sequence given by $a_0 = 1$, $a_{n+1} = \frac{2a_n}{2+\sqrt{4-a_n{}^2}}$. Prove that $\{2^n a_n\}$ is increasing and find its limit.
16. Let $f: [0, \infty] \to \mathbb{R}$ be a differentiable function such that $\int_0^1 f(x)dx = f(1)$ and for every $x \geq 1$ the condition $xf'(x) + f(x-1) = 0$ $holds$. Find $\lim_{x\to\infty} f(x)$.

17. Let $f: [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = \frac{\pi}{4}$. Prove that there exists $x_0 \in (0,1)$ such that $\frac{1}{1+x_0} < f(x_0) < \frac{1}{2x_0}$.

18. Find all differentiable functions f: $\mathbb{R} \to \mathbb{R}$ such that for all x, y $\in \mathbb{R}$ the inequality $f(x+y) \geq 2022^x f(y) + f(x)$ holds and $f(0) = 0$, $f(1) = 1$.

19. A discrete random variable X can assume values x=1,2,3,... with probability 2^{-x}.Show that $P\{|X-2| \geq 2\} \leq \frac{1}{2}$.

20. Find all functions f defined on the positive real numbers and taking positive real values that satisfy the following conditions:

(i)$f(xf(y)) = yf(x)$ for all positive real x and y.

(ii)$f(x) \to 0$ as $x \to +\infty$.